CYCLINE SUBALGEBRAS OF k-GRAPH C*-ALGEBRAS

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Abstract. In this paper, we prove that the cycline subalgbra of a k-graph C*-algebra is maximal abelian, and show when it is a Cartan subalgebra (in the sense of Renault).

1. Introduction

Higher rank graph algebras (or k-graph algebras) have been attracting a lot of attention recently. See, for example, [Rae05] and the references therein. They were first introduced by Kumjian-Pask in 2000 [KP00] in order to generalize directed graph algebras and higher rank Cuntz-Krieger algebras studied by Robertson and Steger [RS99]. For a given k-graph Λ, its graph C*-algebra $\mathrm{C}^*(\Lambda)$ is the universal C*-algebra among Cuntz-Krieger Λ-families.

One of the most important and central topics on k-graph algebras is to determine when a given representation π from $\mathrm{C}^*(\Lambda)$ to a C*-algebra $\mathcal{A}$ is injective. This is closely related to so called "uniqueness theorems" in the literature. There are two such theorems: the *gauge invariant uniqueness theorem* (GIUT) and the *Cuntz-Krieger uniqueness theorem* (CKUT), which have been known for some time. Both GIUT and CKUT conclude that π is injective if and only if its restriction $\pi|_{\mathfrak{D}_\Lambda}$ of π onto the diagonal algebra $\mathfrak{D}_\Lambda$ of $\mathrm{C}^*(\Lambda)$ is injective, under the following conditions: the GIUT requires the existence of an action θ of $\mathbb{T}^k$ on $\mathcal{A}$ such that π is equivariant between θ and the canonical gauge action γ of $\mathbb{T}^k$ on $\mathrm{C}^*(\Lambda)$, while the CKUT requires that Λ is aperiodic.

It is well-known that the aperiodicity is a very stringent condition, and that it is very hard to check (even in single-vertex 2-graphs [DY09a]). Thus, a very important and necessary task is to find a more general version of the CKUT. This has been successfully achieved by Brown-Nagy-Reznikoff recently in [BNR14]. The most natural candidate of $\mathfrak{D}_\Lambda$ in the general case is the so called *cycline subalgebra* $\mathcal{M}_\Lambda$ (whose definition will be precisely given later). Brown-Nagy-Reznikoff proved the following *generalized CKUT*: For *any* row-finite source-free k-graph Λ, a representation π of $\mathrm{C}^*(\Lambda)$ is injective if and only if the restriction $\pi|_{\mathcal{M}_\Lambda}$ is injective.

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Returning to an aperiodic k-graph Λ, it is known that $\mathfrak{D}_\Lambda$ is a MASA (maximal abelian subalgebra) in $\mathrm{C}^*(\Lambda)$, and that there is a faithful conditional expectation from $\mathrm{C}^*(\Lambda)$ onto $\mathfrak{D}_\Lambda$. Besides, in general, for a given abelian C*-subalgebra $\mathcal{B}$ of a C*-algebra $\mathcal{A}$, it is always nice and interesting to know if $\mathcal{B}$ is a MASA in $\mathcal{A}$ and if there is a faithful conditional expectation from $\mathcal{A}$ onto $\mathcal{B}$. So Brown-Nagy-Reznikoff asked the following two natural questions on $\mathcal{M}_\Lambda$ (cf. P. 2591 and P. 2601 in [BNR14]):

Q1. *Is $\mathcal{M}_\Lambda$ a MASA in $\mathrm{C}^*(\Lambda)$?*

Q2. *Is there a faithful conditional expectation from $\mathrm{C}^*(\Lambda)$ onto $\mathcal{M}_\Lambda$?*

Our goal in this note is the following: (a) we answer Q1 affirmatively, (b) we study when Q2 has a positive answer, and so provide a condition which guarantees $\mathcal{M}_\Lambda$ is a Cartan subalgebra in $\mathrm{C}^*(\Lambda)$ (in the sense of Renault [Ren08]).

The remaining of this paper is organized as follows. In the next section, some necessary background is given. Q1 above is completely answered in Section 3. In Section 4 we study when $\mathcal{M}_\Lambda$ is Cartan, and so in particular answer Q2.

2. Preliminaries

In this section, we give some necessary background which will be used later. At the same time, we fix our notation. See [BNR14, CKSS14, KP00, Rae05, Yan14] for more information.

2.1. **k-graphs.** Let $k \geq 1$ be a natural number. Regard $\mathbb{N}^k$ (containing 0) as a small category with one object, and denote its standard generators as $e_1, \ldots, e_k$. A *k-graph* (also known as *rank k graph*, or *higher rank graph*) is a countable small category Λ with a functor $d : \Lambda \to \mathbb{N}^k$ satisfying the following factorization property: Whenever $\xi \in \Lambda$ satisfies $d(\xi) = m + n$, there are unique elements $\eta, \zeta \in \Lambda$ such that $d(\eta) = m$, $d(\zeta) = n$ and $\xi = \eta\zeta$. For $n \in \mathbb{N}^k$, let $\Lambda^n = d^{-1}(n)$, and so Λ^0 is the vertex set of Λ. There are source and range maps $s, r : \Lambda \to \Lambda^0$ such that $r(\xi)\xi s(\xi) = \xi$ for all $\xi \in \Lambda$. For $v \in \Lambda^0$, $v\Lambda = \{\xi \in \Lambda : r(\xi) = v\}$. We say that a k-graph Λ is *row-finite and source-free* if $0 < |v\Lambda^n| < \infty$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

Let

$$\Omega_k = \big\{(m, n) \in \mathbb{N}^k \times \mathbb{N}^k \mid m \leq n\big\}.$$

Define $d, s, r : \Omega_k \to \mathbb{N}^k$ by $d(m, n) = n - m$, $s(m, n) = n$, and $r(m, n) = m$. One can check that Ω_k is a row-finite and source-free k-graph.

Let Λ and Γ be two k-graphs. A *k-graph morphism* between Λ and Γ is a functor $x : \Lambda \to \Gamma$ such that $d_\Gamma(x(\lambda)) = d_\Lambda(\lambda)$ for all $\lambda \in \Lambda$. The *infinite path space of* Λ is defined as

$$\Lambda^\infty = \big\{x : \Omega_k \to \Lambda \mid x \text{ is a } k\text{-graph morphism}\big\}.$$

If Λ is row-finite and source-free, it is often useful to think of every element of Λ^∞ as an infinite path, which contains infinitely many edges of degree e_i

for each $i \in \{1, \dots, k\}$. For $x \in \Lambda^\infty$ and $n \in \mathbb{N}^k$, there is a unique element $\sigma^n(x) \in \Lambda^\infty$ defined by

$$\sigma^n(x)(q,r) = x(n+q, n+r).$$

That is, σ^n is a shift map on Λ^∞. If $\mu \in \Lambda$ and $x \in s(\mu)\Lambda^\infty$, then μx is defined to be the unique infinite path such that $\mu x(0,n) = \mu \cdot x(0, n - d(\mu))$ for any $n \in \mathbb{N}^k$ with $n \geq d(\mu)$. If $\sigma^m(x) = \sigma^n(x)$ for some $m \neq n$ in $\mathbb{N}^k$, x is said to be (eventually) *periodic*.

Definition 2.1. A k-graph Λ is said to be *periodic* if there is $v \in \Lambda^0$ such that every $x \in v\Lambda^\infty$ is periodic. Otherwise, Λ is called *aperiodic*.

2.2. **k-graph C*-algebras.** For a given row-finite and source-free k-graph Λ, we associate to it a universal C*-algebra $\mathrm{C}^*(\Lambda)$ as follows.

Definition 2.2. Let Λ be a row-finite and source-free k-graph. A *Cuntz-Krieger Λ-family* in a C*-algebra $\mathcal{A}$ is a family $\{S_\lambda : \lambda \in \Lambda\}$ in $\mathcal{A}$ such that

(CK1) $\{S_v \mid v \in \Lambda^0\}$ is a set of mutually orthogonal projections;
(CK2) $S_\mu S_\nu = S_{\mu\nu}$ whenever $s(\mu) = r(\nu)$;
(CK3) $S_\mu^* S_\nu = \delta_{\mu,\nu}\, S_{s(\mu)}$ for all $\mu, \nu \in \Lambda$ with $d(\mu) = d(\nu)$;
(CK4) $S_v = \sum_{\lambda \in v\Lambda^n} S_\lambda S_\lambda^*$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

The *k-graph C*-algebra* $\mathrm{C}^*(\Lambda)$ is the universal C*-algebra among Cuntz-Krieger Λ-families. In this paper, we use $\{s_\mu \mid \mu \in \Lambda\}$ to denote the universal Cuntz-Krieger Λ-family of $\mathrm{C}^*(\Lambda)$.

It is known that

$$\mathrm{C}^*(\Lambda) = \overline{\mathrm{span}}\{s_\mu s_\nu^* : \mu, \nu \in \Lambda\}.$$

By the universal property of $\mathrm{C}^*(\Lambda)$, there is a natural gauge action γ of $\mathbb{T}^k$ on $\mathrm{C}^*(\Lambda)$ defined by

$$\gamma_t(s_\lambda) = t^{d(\lambda)} s_\lambda \quad \text{for all} \quad t \in \mathbb{T}^k,\ \lambda \in \Lambda.$$

Here $t^n = t_1^{n_1} \cdots t_k^{n_k}$ for all $n = (n_1, \dots, n_k) \in \mathbb{Z}^k$. Averaging over γ gives a faithful conditional expectation Φ from $\mathrm{C}^*(\Lambda)$ onto the fixed point algebra $\mathrm{C}^*(\Lambda)^\gamma$, known as the *core* of $\mathrm{C}^*(\Lambda)$. It turns out that $\mathrm{C}^*(\Lambda)^\gamma$ is an AF algebra and

$$\mathfrak{F}_\Lambda := \mathrm{C}^*(\Lambda)^\gamma = \overline{\mathrm{span}}\{s_\mu s_\nu^* : d(\mu) = d(\nu)\}.$$

For sake of simplicity, put

$$P_\mu := s_\mu s_\mu^* \quad \text{for all} \quad \mu \in \Lambda.$$

The *diagonal algebra* $\mathfrak{D}_\Lambda$ of $\mathrm{C}^*(\Lambda)$ is defined as

$$\mathfrak{D}_\Lambda = \overline{\mathrm{span}}\{s_\mu s_\mu^* : \mu \in \Lambda\} = \overline{\mathrm{span}}\{P_\mu : \mu \in \Lambda\},$$

which is a MASA in $\mathfrak{F}_\Lambda$, but, generally not a MASA in $\mathrm{C}^*(\Lambda)$.

For each $n = (n_1, \ldots, n_k) \in \mathbb{Z}^k$, define a mapping Φ_n on $\mathrm{C}^*(\Lambda)$ via

$$\Phi_n(x) = \int_{\mathbb{T}^k} t^{-n} \gamma_t(x) dt \quad \text{for all} \quad x \in \mathrm{C}^*(\Lambda).$$

Then Φ_n acts on the standard generators via

$$\Phi_n(s_\mu s_\nu^*) = \begin{cases} s_\mu s_\nu^*, & \text{if} \quad d(\mu) - d(\nu) = n, \\ 0, & \text{otherwise.} \end{cases}$$

So $\mathfrak{F}_\Lambda$ coincides with $\operatorname{Ran} \Phi_0$, and $\operatorname{Ran} \Phi_n$ is spanned by the standard generators in $\mathrm{C}^*(\Lambda)$ of "degree n". Also, as directed graph algebras [HPP05], every $x \in \mathrm{C}^*(\Lambda)$ has a (unique) formal series

$$x \sim \sum_{n \in \mathbb{Z}^k} \Phi_n(x),$$

which is Abel summable (refer to [Taylo] for information on Abel summable). It is often useful heuristically to work directly with the series of x.

Row-finite and source-free k-graph C*-algebras can be also constructed via second countable, étale locally compact groupoids

$$\mathcal{G}_\Lambda = \big\{ (x, m-n, y) \in \Lambda^\infty \times \mathbb{Z}^k \times \Lambda^\infty : \sigma^m(x) = \sigma^n(y) \big\}$$

(cf. [KP00]). The following facts are well-known: A basis for the topology of $\mathcal{G}_\Lambda$ is given by the open compact cylinder sets

$$Z(\alpha, \beta) = \big\{ (\alpha x, d(\alpha) - d(\beta), \beta x) : x \in s(\alpha)\Lambda^\infty \big\},$$

where $\alpha, \beta \in \Lambda$ with $s(\alpha) = s(\beta)$; $\mathcal{G}_\Lambda$ is amenable; and $\mathrm{C}^*(\Lambda) \cong \mathrm{C}^*(\mathcal{G}_\Lambda) \cong \mathrm{C}^*_r(\mathcal{G}_\Lambda)$. From [Ren80], $\mathrm{C}^*(\mathcal{G}_\Lambda)$ consists of some elements of $\mathrm{C}_0(\mathcal{G}_\Lambda)$, the continuous functions on $\mathcal{G}_\Lambda$ vanishing at infinity. But also notice that $\mathrm{C}^*(\mathcal{G}_\Lambda)$ contains $\mathrm{C}_c(\mathcal{G}_\Lambda)$, the continuous functions on $\mathcal{G}_\Lambda$ with compact support.

2.3. **Periodicity.** Let Λ be a row-finite and source-free k-graph. Define an equivalence relation $\sim$ on Λ as follows:

$$\mu \sim \nu \Longleftrightarrow s(\mu) = s(\nu) \text{ and } \mu x = \nu x \text{ for all } x \in s(\mu)\Lambda^\infty. \tag{1}$$

If $\mu \sim \nu$, obviously one also has $r(\mu) = r(\nu)$ automatically. So $\sim$ respects sources and ranges.

Associate to the equivalence relation $\sim$ an important set – the *periodicity* $\operatorname{Per}\Lambda$ *of* Λ:

$$\operatorname{Per}\Lambda = \big\{ d(\mu) - d(\nu) : \xi, \eta \in \Lambda,\ \mu \sim \nu \big\} \subseteq \mathbb{Z}^k.$$

In general, $\operatorname{Per}\Lambda$ is a subset of $\mathbb{Z}^k$ containing 0. Furthermore, Λ is aperiodic if and only if $\operatorname{Per}\Lambda = \{0\}$ (cf., e.g., [Yan14]).

The subalgebra we are particularly interested in here is

$$\mathcal{M}_\Lambda = \mathrm{C}^*(s_\mu s_\nu^* : \mu \sim \nu \in \Lambda),$$

which plays a vital role in this paper. $\mathcal{M}_\Lambda$ is called the *cycline subalgebra of* $\mathrm{C}^*(\Lambda)$ in [BNR14], since it is related to generalized cycles introduced in [ES12]. Actually, it is defined in terms of cycline pairs in [BNR14]. But it

is the same as the one defined above by [BNR14, Proposition 4.1] and the definition of $\sim$ in (1). Clearly, $\mathcal{M}_\Lambda$ contains $\mathfrak{D}_\Lambda$. Furthermore, $\mathcal{M}_\Lambda = \mathfrak{D}_\Lambda$ if and only if Λ is aperiodic by the characterization of aperiodicity mentioned above. It is also shown in [BNR14] that the relative commutant $\mathfrak{D}_\Lambda'$ of $\mathfrak{D}_\Lambda$ in $\mathrm{C}^*(\Lambda)$ is abelian, and that $\mathcal{M}_\Lambda \subseteq \mathcal{M}_\Lambda' = \mathfrak{D}_\Lambda'$.

Let us finish off this section by the following conventions.

Conventions. Throughout the rest of this paper,

all k-graphs are assumed to be row-finite and source-free

without any further mention.

For simplicity, we write $\mathfrak{D}_\Lambda'$ to really mean the relative commutant

$$\mathfrak{D}_\Lambda' = \big\{A \in \mathrm{C}^*(\Lambda) : AD = DA \text{ for all } D \in \mathfrak{D}_\Lambda\big\}.$$

3. Cycline subalgebras are MASA

Let Λ be a k-graph, and $\mathcal{M}_\Lambda$ be the cycline subalgebra of $\mathrm{C}^*(\Lambda)$. Recall from [BNR14] that

$$\mathcal{M}_\Lambda = \mathrm{C}^*(s_\mu s_\nu^* : \mu \sim \nu) = \overline{\mathrm{span}}\big\{s_\mu s_\nu^* : \mu \sim \nu\big\}.$$

Our main goal in this section is to prove $\mathcal{M}_\Lambda = \mathfrak{D}_\Lambda'$, which in particular affirmatively answers Q1 mentioned in Introduction: $\mathcal{M}_\Lambda$ is always a MASA. But four auxiliary lemmas are needed. The first one is directly from [BNR14].

Lemma 3.1. [BNR14, Proposition 4.1] *Let $\mu, \nu \in \Lambda$ with $s(\mu) = s(\nu)$. The following are equivalent:*

(i) *$s_\mu s_\nu^*$ is normal and commutes with $\mathfrak{D}_\Lambda$;*
(ii) *$\mu \sim \nu$.*

By Lemma 3.1, one has $\mathcal{M}_\Lambda \subseteq \mathfrak{D}_\Lambda'$. So in order to prove $\mathcal{M}_\Lambda = \mathfrak{D}_\Lambda'$, it is sufficient to verify $\mathfrak{D}_\Lambda' \subseteq \mathcal{M}_\Lambda$. Our first step is to prove that the standard generators in $\mathfrak{D}_\Lambda'$ belong to $\mathcal{M}_\Lambda$. The following gives a strengthened version of Lemma 3.1, which will be very useful in what follows.

Lemma 3.2. *Let $\mu, \nu \in \Lambda$ with $s(\mu) = s(\nu)$. Then*

$$s_\mu s_\nu^* \in \mathfrak{D}_\Lambda' \Longleftrightarrow \mu \sim \nu.$$

Proof. By Lemma 3.1, it is enough to prove that $s_\mu s_\nu^* \in \mathfrak{D}_\Lambda'$ implies that $s_\mu s_\nu^*$ is automatically normal, namely,

$$s_\mu s_\mu^* = s_\nu s_\nu^*, \text{ i.e., } P_\mu = P_\nu.$$

Assume now that $\mu, \nu \in \Lambda$ with $s(\mu) = s(\nu)$ satisfies $s_\mu s_\nu^* \in \mathfrak{D}'_\Lambda$. Then one has the following implications:

$$\begin{aligned}
& s_\mu s_\mu^* \cdot s_\mu s_\nu^* = s_\mu s_\nu^* \cdot s_\mu s_\mu^* \\
\Rightarrow\ & s_\mu s_\nu^* = s_\mu s_\nu^* s_\mu s_\mu^* \\
\Rightarrow\ & s_\mu^* s_\mu s_\nu^* = s_\mu^* s_\mu s_\nu^* s_\mu s_\mu^* \\
\Rightarrow\ & s_{s(\mu)} s_\nu^* = s_{s(\mu)} s_\nu^* P_\mu \\
\Rightarrow\ & s_\nu^* = s_\nu^* P_\mu \\
\Rightarrow\ & s_\nu s_\nu^* = s_\nu s_\nu^* P_\mu \\
\Rightarrow\ & P_\nu = P_\nu P_\mu.
\end{aligned}$$

Clearly $s_\mu s_\nu^* \in \mathfrak{D}'_\Lambda$ implies $s_\nu s_\mu^* \in \mathfrak{D}'_\Lambda$. So switching μ and ν in the above process gives $P_\mu = P_\mu P_\nu$. Hence $P_\mu = P_\nu$ as $P_\mu P_\nu = P_\nu P_\mu$. This ends our proof. ∎

Roughly speaking, the next lemma says that, for our purpose, it is enough to consider the elements of $\mathfrak{D}'_\Lambda$ of degree $n \in \mathbb{Z}^k$.

Lemma 3.3. *Let $A \in \mathrm{C}^*(\Lambda)$. Then $A \in \mathfrak{D}'_\Lambda$ if and only if $\Phi_n(A) \in \mathfrak{D}'_\Lambda$ for all $n \in \mathbb{Z}^k$.*

Proof. It suffices to show the "only if" part. Let $D \in \mathfrak{D}_\Lambda$. Then for all $n \in \mathbb{Z}^k$ one has

$$\begin{aligned}
\Phi_n(A)D &= \int_{\mathbb{T}^k} t^{-n}\gamma_t(A)\,dt\,D \\
&= \int_{\mathbb{T}^k} t^{-n}\gamma_t(A)\gamma_t(D)\,dt \quad (\text{as } \gamma_t(D) = D) \\
&= \int_{\mathbb{T}^k} t^{-n}\gamma_t(AD)\,dt \\
&= \int_{\mathbb{T}^k} t^{-n}\gamma_t(DA)\,dt \quad (\text{as } A \in \mathfrak{D}'_\Lambda) \\
&= D\int_{\mathbb{T}^k} t^{-n}\gamma_t(A)\,dt \quad (\text{as } \gamma_t(D) = D) \\
&= D\Phi_n(A).
\end{aligned}$$

This proves $\Phi_n(A) \in \mathfrak{D}'_\Lambda$. ∎

It turns out that, for $n \in \mathbb{Z}^k$, any element in $\mathfrak{D}'_\Lambda$ of degree n in a certain "canonical" form is very special: It essentially has only one term. This is not surprising if one keeps the uniqueness result [Yan14, Lemma 4.1] in mind.

Lemma 3.4. *Let $m, n \in \mathbb{N}^k$ and*

$$A = \sum_{d(\mu)=m,\, d(\nu)=n,\, s(\mu)=s(\nu)} a_{\mu,\nu}\, s_\mu s_\nu^* \in \mathrm{C}^*(\Lambda).$$

Then the following hold true:

(i) *If $A \in \mathfrak{D}'_\Lambda$, then, for each $\mu \in \Lambda^m$, there is a unique $\nu \in \Lambda^n$ such that $a_{\mu,\nu} \neq 0$.*
(ii) *If $A \in \mathfrak{D}'_\Lambda$, then, for each $\nu \in \Lambda^n$, there is a unique $\mu \in \Lambda^m$ such that $a_{\mu,\nu} \neq 0$.*

Proof. It suffices to show (i), since once (i) is established (ii) follows by applying (i) to A^*.

Let

$$A = \sum_{d(\mu)=m,\, d(\nu)=n,\, s(\mu)=s(\nu)} a_{\mu,\nu}\, s_\mu s_\nu^* \in \mathfrak{D}'_\Lambda.$$

Notice that $s_\mu s_\nu^* \neq 0$ as $s(\mu) = s(\nu)$. Assume that $\mu_0 \in \Lambda^m$ is such that $a_{\mu_0,\nu_0} \neq 0$ for some $\nu_0 \in \Lambda^n$. Then we must show the uniqueness of ν_0.

Since $A \in \mathfrak{D}'_\Lambda$, we have

$$s_{\mu_0} s_{\mu_0}^* A = A s_{\mu_0} s_{\mu_0}^*.$$

Multiplying $s_{\mu_0}^*$ from the left at both sides in the above identity induces

$$s_{\mu_0}^* A = s_{\mu_0}^* A s_{\mu_0} s_{\mu_0}^*$$

as s_{μ_0} is a partial isometry. Then we expand it using the formula of A and then calculate both sides to obtain

$$\sum_{\nu \in \Lambda^n s(\mu_0)} a_{\mu_0,\nu}\, s_\nu^* = \sum_{\nu \in \Lambda^n s(\mu_0)} a_{\mu_0,\nu}\, s_\nu^* s_{\mu_0} s_{\mu_0}^*. \tag{2}$$

Multiplying s_{ν_0} from right at both sides of (2) and using (CK3) yields

$$\begin{aligned}
& a_{\mu_0,\nu_0} s_{\nu_0}^* s_{\nu_0} = \sum_{\nu \in \Lambda^n s(\mu_0)} a_{\mu_0,\nu}\, s_\nu^* s_{\mu_0} s_{\mu_0}^* s_{\nu_0} \\
\Rightarrow\ & a_{\mu_0,\nu_0} s_{\nu_0} \cdot s_{\nu_0}^* s_{\nu_0} \cdot s_{\nu_0}^* = \sum_{\nu \in \Lambda^n s(\mu_0)} a_{\mu_0,\nu}\, s_{\nu_0} \cdot s_\nu^* s_{\mu_0} s_{\mu_0}^* s_{\nu_0} \cdot s_{\nu_0}^* \\
\Rightarrow\ & a_{\mu_0,\nu_0} s_{\nu_0} s_{\nu_0}^* = \sum_{\nu \in \Lambda^n s(\mu_0)} a_{\mu_0,\nu}\, s_{\nu_0} s_\nu^* s_{\nu_0} s_{\nu_0}^* s_{\mu_0} s_{\mu_0}^* \quad (\text{as } P_{\mu_0} P_{\nu_0} = P_{\nu_0} P_{\mu_0}) \\
\Rightarrow\ & a_{\mu_0,\nu_0} s_{\nu_0} s_{\nu_0}^* = a_{\mu_0,\nu_0}\, s_{\nu_0} s_{\nu_0}^* s_{\mu_0} s_{\mu_0}^* \\
\Rightarrow\ & P_{\nu_0} = P_{\nu_0} P_{\mu_0} \quad (\text{by (CK3) and } a_{\mu_0,\nu_0} \neq 0).
\end{aligned}$$

Completely similar reasoning (by considering A^* instead of A) gives $P_{\mu_0} = P_{\mu_0} P_{\nu_0}$. Therefore we so far have shown that, for *any* μ_0, ν_0 such that $a_{\mu_0,\nu_0} \neq 0$, we have

$$P_{\mu_0} = P_{\nu_0}. \tag{3}$$

From (2) one also induces that

$$\left(\sum_{\nu\in\Lambda^n s(\mu_0)} a_{\mu_0,\nu}\, s_\nu^*\right)\left(\sum_{\nu\in\Lambda^n s(\mu_0)} a_{\mu_0,\nu}\, s_\nu^*\right)^*$$
$$=\left(\sum_{\nu\in\Lambda^n s(\mu_0)} a_{\mu_0,\nu}\, s_\nu^* s_{\mu_0} s_{\mu_0}^*\right)\left(\sum_{\nu\in\Lambda^n s(\mu_0)} a_{\mu_0,\nu}\, s_\nu^* s_{\mu_0} s_{\mu_0}^*\right)^*.$$

Expanding both sides and then using (3) and (CK3), we obtain

$$\begin{aligned}\sum_{\nu\in\Lambda^n s(\mu_0)} |a_{\mu_0,\nu}|^2 s_{s(\nu)} &= \sum_{\nu_1,\,\nu_2\in\Lambda^n s(\mu_0)} a_{\mu_0,\nu_1}\overline{a}_{\mu_0,\nu_2}\, s_{\nu_1}^* s_{\mu_0} s_{\mu_0}^* s_{\mu_0} s_{\mu_0}^* s_{\nu_2}\\ &= \sum_{\nu_1,\,\nu_2\in\Lambda^n s(\mu_0)} a_{\mu_0,\nu_1}\overline{a}_{\mu_0,\nu_2}\, s_{\nu_1}^* \cdot s_{\mu_0} s_{\mu_0}^* \cdot s_{\nu_2}\\ &= \sum_{\nu_1,\,\nu_2\in\Lambda^n s(\mu_0)} a_{\mu_0,\nu_1}\overline{a}_{\mu_0,\nu_2}\, s_{\nu_1}^* \cdot s_{\nu_0} s_{\nu_0}^* \cdot s_{\nu_2} \quad \text{(by (3))}\\ &= |a_{\mu_0,\nu_0}|^2 s_{\nu_0}^* s_{\nu_0} s_{\nu_0}^* s_{\nu_0} \quad \text{(from (CK3))}\\ &= |a_{\mu_0,\nu_0}|^2 s_{s(\nu_0)}.\end{aligned}$$

Therefore, $a_{\mu_0,\nu} = 0$ for all $\nu \neq \nu_0$, proving the uniqueness of ν_0. ∎

We are now ready to prove our main result in this section.

Theorem 3.5. *Let Λ be a k-graph, and $\mathcal{M}_\Lambda$ be the cycline algebra of $\mathrm{C}^*(\Lambda)$. Then $\mathcal{M}_\Lambda = \mathfrak{D}_\Lambda'$. In particular, $\mathcal{M}_\Lambda$ is a MASA in $\mathrm{C}^*(\Lambda)$.*

Proof. We first prove $\mathcal{M}_\Lambda = \mathfrak{D}_\Lambda'$. The inclusion $\mathcal{M}_\Lambda \subseteq \mathfrak{D}_\Lambda'$ is clearly from Lemma 3.1 (ii) $\Rightarrow$ (i). So we must show $\mathfrak{D}_\Lambda' \subseteq \mathcal{M}_\Lambda$ in what follows.

It is easy to verify that $\mathfrak{D}_\Lambda'$ is a gauge invariant $\mathfrak{D}_\Lambda$-bimodule. Using an argument similar to [HPP05, Theorem 3.1] (also cf. [Hop05]), one can show that $\mathfrak{D}_\Lambda'$ is generated by the standard generators $s_\mu s_\nu^*$ which it contains. Let $\mathfrak{A}$ be the 'algebraic' part of $\mathfrak{D}_\Lambda'$. That is, $\mathfrak{A}$ is the algebra of the *finite* linear span of those standard generators. Then $\mathfrak{A}$ is dense in $\mathfrak{D}_\Lambda'$. So, for our purpose, it suffices to show that $\mathfrak{A} \subseteq \mathcal{M}_\Lambda$. For this, let $A \in \mathfrak{A}$. By Lemma 3.3, without loss of generality, let us assume that A is of degree n for some $n \in \mathbb{Z}^k$: $A = \Phi_n(A)$. Recall that A is just a (finite) linear combination of some generators $s_\mu s_\nu^*$'s. Using the defect-free property (CK4), one can now simply write A as follows:

$$A = \sum_{d(\mu)=m,\, d(\nu)=n,\, s(\mu)=s(\nu)} a_{\mu,\nu}\, s_\mu s_\nu^*,$$

where m, n are two fixed elements in $\mathbb{N}^k$, and $a_{\mu,\nu} \neq 0$. It then follows from Lemma 3.4 for a fixed $\nu \in \Lambda^n$, there is a unique $\mu \in \Lambda^m$ such that $a_{\mu,\,\nu} \neq 0$. Then

$$A s_\nu s_\nu^* = \left(\sum a_{\mu,\nu}\, s_\mu s_\nu^*\right) s_\nu s_\nu^* = a_{\mu,\nu} s_\mu s_\nu^*.$$

Clearly $As_\nu s_\nu^* \in \mathfrak{D}'_\Lambda$ as $A \in \mathfrak{D}'_\Lambda$ and $s_\nu s_\nu^* \in \mathfrak{D}_\Lambda$. So $s_\mu s_\nu^* \in \mathfrak{D}'_\Lambda$. By Lemma 3.2, $\mu \sim \nu$, which implies that $A \in \mathcal{M}_\Lambda$. Therefore, $\mathcal{M}_\Lambda = \mathfrak{D}'_\Lambda$.

The second part of the theorem follows immediately, since it is known that $\mathfrak{D}'_\Lambda$ is abelian and $\mathcal{M}'_\Lambda = \mathfrak{D}'_\Lambda$ ([BNR14, Proposition 7.3]). ∎

By Theorem 3.5, one can easily recover the following characterization of aperiodicity, one of the main theorems in [Hop05].

Corollary 3.6. *A k-graph Λ is aperiodic if and only if the diagonal algebra $\mathfrak{D}_\Lambda$ is a MASA in $\mathrm{C}^*(\Lambda)$.*

Proof. $\mathfrak{D}_\Lambda$ is a MASA in $\mathrm{C}^*(\Lambda)$, if and only if $\mathfrak{D}'_\Lambda = \mathfrak{D}_\Lambda$, if and only if $\mathcal{M}_\Lambda = \mathfrak{D}_\Lambda$ by Theorem 3.5, if and only if no $\mu \neq \nu$ such that $\mu \sim \nu$ by definition of $\mathcal{M}_\Lambda$, if and only if $\operatorname{Per}\Lambda = \{0\}$ by definition of $\operatorname{Per}\Lambda$, if and only if Λ is aperiodic (see, e.g., [RS07] or [Yan14]). ∎

4. When are Cyline subalgebras Cartan

Let $\mathcal{B}$ be an abelian C*-subalgebra of a given C*-algebra $\mathcal{A}$. Recall from [Ren08] that $\mathcal{B}$ is a *Cartan subalgebra* in $\mathcal{A}$ if the following properties hold:

- (Ci) $\mathcal{B}$ contains an approximate unit in $\mathcal{A}$;
- (Cii) $\mathcal{B}$ is a MASA;
- (Ciii) $\mathcal{B}$ is regular, i.e., the normalizer set $N(\mathcal{B}) = \{x \in \mathcal{A} : x\mathcal{B}x^* \cup x^*\mathcal{B}x \subseteq \mathcal{B}\}$ generates $\mathcal{A}$;
- (Civ) there is a faithful conditional expectation $\mathcal{E}$ from $\mathcal{A}$ onto $\mathcal{B}$.

Let Λ be a k-graph. In this section, we prove that the cycline subalgebra $\mathcal{M}_\Lambda$ of $\mathrm{C}^*(\Lambda)$ is Cartan under the condition that the (bimodule) spectrum of $\mathcal{M}_\Lambda$ (the definition will be given later) is closed, which is used to obtain property (Civ).

Proposition 4.1. *Let Λ be a k-graph, and $\mathcal{M}_\Lambda$ be the cycline subalgebra of $\mathrm{C}^*(\Lambda)$. Then $\mathcal{M}_\Lambda$ is regular.*

Proof. Since $\mathrm{C}^*(\Lambda)$ is generated by its standard generators $s_\alpha s_\beta^*$'s ($\alpha, \beta \in \Lambda$), it suffices to show that every $s_\alpha s_\beta^*$ is a normalizer of $\mathcal{M}_\Lambda$. But $\mathcal{M}_\Lambda$ is generated by $s_\mu s_\nu^*$ with $\mu \sim \nu$. Hence, one only needs to show that

$$s_\alpha s_\beta^* s_\mu s_\nu^* s_\beta s_\alpha^* \in \mathcal{M}_\Lambda.$$

To this end, let us assume that

$$s_\beta^* s_\mu = \sum_{\beta\mu' = \mu\beta'} s_{\mu'} s_{\beta'}^* \quad \text{and} \quad s_\nu^* s_\beta = \sum_{\beta\nu' = \nu\beta''} s_{\beta''} s_{\nu'}^*.$$

Then

$$
\begin{aligned}
& s_\alpha s_\beta^* s_\mu s_\nu^* s_\beta s_\alpha^* \\
=& s_\alpha \left(\sum_{\beta\mu'=\mu\beta'} s_{\mu'} s_{\beta'}^* \right) \left(\sum_{\beta\nu'=\nu\beta''} s_{\beta''} s_{\nu'}^* \right) s_\alpha^* \\
=& s_\alpha \left(\sum_{\beta\mu'=\mu\beta',\, \beta\nu'=\nu\beta'} s_{\mu'} s_{\nu'}^* \right) s_\alpha^* \quad \big(\text{as } d(\beta')=d(\beta'') \Rightarrow s_{\beta'}^* s_{\beta''} = \delta_{\beta',\beta''}\, s_{s(\beta')}\big) \\
=& \sum_{\beta\mu'=\mu\beta',\, \beta\nu'=\nu\beta'} s_{\alpha\mu'} s_{\alpha\nu'}^*. \qquad (4)
\end{aligned}
$$

Clearly $\mu\beta' \sim \nu\beta'$ as $\mu \sim \nu$. It follows from $\beta\mu' = \mu\beta'$, $\beta\nu' = \nu\beta'$ that $\beta\mu' \sim \beta\nu'$. This easily implies $\mu' \sim \nu'$ and so $\alpha\mu' \sim \alpha\nu'$. Therefore one has $s_\alpha s_\beta^* s_\mu s_\nu^* s_\beta s_\alpha^* \in \mathcal{M}_\Lambda$ from (4). This ends our proof. ∎

In what follows, we identify $\mathrm{C}^*(\Lambda)$ with $\mathrm{C}^*(\mathcal{G}_\Lambda)$ under the isomorphism mapping $s_\alpha s_\beta^* \mapsto 1_{Z(\alpha,\beta)}$, where $1_{Z(\alpha,\beta)}$ is the characteristic function of the cylinder set $Z(\alpha,\beta)$ (cf. [KP00]). Then the diagonal algebra $\mathfrak{D}_\Lambda$ is identified as $\mathrm{C}_0(\mathcal{G}_\Lambda^{(0)})$, where $\mathcal{G}_\Lambda^{(0)}$ is the unit space of $\mathcal{G}_\Lambda$; and $\mathcal{M}_\Lambda$ is generated by $1_{Z(\mu,\nu)}$'s with $\mu \sim \nu \in \Lambda$. Evidently, $\mathcal{M}_\Lambda$ is a $\mathfrak{D}_\Lambda$-bimodule. By definition [Hop05], its (bimodule) *spectrum* is

$$\sigma(\mathcal{M}_\Lambda) = \big\{(x,n,y) \in \mathcal{G}_\Lambda : f(x,n,y) \neq 0 \text{ for some } f \in \mathcal{M}_\Lambda\big\}.$$

Clearly, $\sigma(\mathcal{M}_\Lambda)$ is always an open subset of $\mathcal{G}_\Lambda$.

Proposition 4.2. *Let Λ be a k-graph, and $\mathcal{M}_\Lambda$ be the cycline subalgebra of $\mathrm{C}^*(\Lambda)$. If $\sigma(\mathcal{M}_\Lambda)$ is closed in $\mathcal{G}_\Lambda$, then there is a faithful conditional expectation from $\mathrm{C}^*(\Lambda)$ onto $\mathcal{M}_\Lambda$.*

Proof. It is easy to see that $\mathcal{M}_\Lambda$ is a gauge invariant $\mathfrak{D}_\Lambda$-bimodule. It then follows from [Hop05, Spectral Theorem for Bimodules on P. 997] that one has

$$\mathcal{M}_\Lambda = \big\{f \in \mathrm{C}^*(\mathcal{G}_\Lambda) : f(x,n,y) = 0 \text{ for all } (x,n,y) \notin \sigma(\mathcal{M}_\Lambda)\big\}.$$

Since $\sigma(\mathcal{M}_\Lambda)$ is closed in $\mathcal{G}_\Lambda$, one now can define the restriction mapping $\mathcal{E}$: $\mathrm{C}^*(\mathcal{G}_\Lambda) \to \mathcal{M}_\Lambda$ via $\mathcal{E}(f) = f|_{\sigma(\mathcal{M}_\Lambda)}$ for all $f \in \mathrm{C}^*(\mathcal{G}_\Lambda)$. Clearly, $\mathcal{E}$ is a linear idempotent. Using the definition of the norm in $\mathrm{C}^*_r(\mathcal{G}_\Lambda)$,[1] one can see that $\mathcal{E}$ is also contractive. Furthermore, it is known that the restriction mapping

[1]Recall that the reduced C*-norm on $\mathrm{C}_c(\mathcal{G}_\Lambda)$ is given by

$$\|f\| = \sup\big\{\|\lambda_u(f)\| : u \in \mathcal{G}_\Lambda^{(0)}\big\} \text{ for all } f \in \mathrm{C}_c(\mathcal{G}_\Lambda),$$

where λ_u is the regular representation of $\mathrm{C}_c(\mathcal{G}_\Lambda)$ on $\ell^2(s^{-1}(u))$:

$$\lambda_u(f)\xi(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)\xi(\beta) \text{ for all } u \in \mathcal{G}^{(0)},\ \xi \in \ell^2(s^{-1}(u)) \text{ and } \gamma \in s^{-1}(u).$$

E from $\mathrm{C}^*(\mathcal{G}_\Lambda)$ to $\mathfrak{D}_\Lambda$ yields a faithful conditional expectation onto $\mathfrak{D}_\Lambda$ (see, e.g., [BCS14, Ren08, Tho10]). In particular, $\|E\| = 1$. Then it follows from $E = E \circ \mathcal{E}$ that $\mathcal{E}$ has norm 1. Thus $\mathcal{E}$ is a conditional expectation onto $\mathcal{M}_\Lambda$ ([Tom57] or [Bla06, II.6.10]). Moreover, the faithfulness of E and $E = E \circ \mathcal{E}$ imply that $\mathcal{E}$ is faithful too. This ends the proof. ∎

Let us remark that the above proposition could be also proved by modifying the proof of [Tho10, Lemma 2.21]. Notice that the mapping Q in (2.16) there is our restriction mapping. As our condition of $\sigma(\mathcal{M}_\Lambda)$ being closed, the discreteness of the isotropy group guarantees that Q is well-defined. Also, one has $\mathcal{M}_\Lambda = \mathrm{C}^*_r(\sigma(\mathcal{M}_\Lambda))$ by [Tho10, Lemma2.10].

Theorem 4.3. *Let Λ be a k-graph. Suppose that the (bimodule) spectrum $\sigma(\mathcal{M}_\Lambda)$ is closed in $\mathcal{G}_\Lambda$. Then $\mathcal{M}_\Lambda$ is a Cartan subalgebra in $\mathrm{C}^*(\Lambda)$.*

Proof. The proof is now immediate: Property (Ci) follows from [BCS14, Lemma 2.1] (also cf. the proof of [Tho10, Theorem 2.23]); properties (Cii), (Ciii) and (Civ) are from Theorem 3.5, Proposition 4.1 and Proposition 4.2, respectively. ∎

Remark 4.4. It is probably worthwhile to mention that Q1 and Q2 mentioned in Introduction were successfully attacked for directed graphs (i.e., 1-graphs) in [NR12]. Unfortunately, the methods there can not be applied to k-graphs, due to the complexity caused by periodicity in higher dimensional cases. Notice that Theorem 4.3 is proved in [Yan14] for a special class of k-graphs as an application of an embedding theorem.

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Note added in proof. After this paper was circulated, the main results of this paper were generalized by Brown-Nagy-Reznikoff-Sims-Williams in the recent paper [BNRSW15] by using completely different approaches.

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